Lesson

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In this Lesson
- The Analytic Hierarchy Process.
- The basic principles.
- Some theory and mathematics behind AHP.

Duration : 100 minutes

The Analytic Hierarchy Process

The ***Analytic Hierarchy Process (AHP)*** is due to ***Saaty*** (1980) and quite often is referred to, as the ***Saaty method***. It is popular and widely used, in decision making and in a wide range of applications. Saaty, in his book, describes case applications ranging from the choice of a school for his son, through to the planning of transportation systems for the Sudan. There is much more to the AHP than we have space for but we will try to cover the most easily used aspects of it.

The AHP deals with problems of the following type. A firm wishes to buy one new piece of equipment of a certain type and has four aspects in mind which will govern its purchasing choice: expense (E), operability (O), reliability (R), and flexibility (F). Competing manufacturers of that equipment have offered three options, X, Y and Z. The firm's engineers have looked at these options and decided that X is cheap and easy to operate but is not very reliable and could not easily be adapted to other uses. Y is somewhat more expensive, is reasonably easy to operate, and is very reliable but not very adaptable. Finally, Z is very expensive, not easy to operate, is a little less reliable than Y but is claimed by the manufacturer to have a wide range of alternative uses. Each of X, Y and Z will satisfy the firms requirements to differing extents so which, overall, best meets this firms needs? This is clearly an important and common class of problem and the AHP has numerous applications but also some limitations which will be discussed at the end of this help section.

The basic principles

The mathematics of the AHP and the calculation techniques are explained in the section "The AHP calculations" but its essence is to construct a matrix expressing the relative values of a set of attributes. For example, what is the relative importance to the management of this firm of the cost of equipment as opposed to its ease of operation? They are asked to choose whether cost is very much more important, rather more important, and as important, and so on down to very much less important, than operability. Each of these judgments is assigned a number on a scale. One common scale (adapted from Saaty) is the one shown in ***Table 1***.

Intensity of importance	Definition	Explanation
1	Equal importance	Two factors contribute equally to the objective.
3	Somewhat more important	Experience and judgment slightly favor one over the other.
5	Much more important	Experience and judgment strongly favor one over the other.
7	Very much more important	Experience and judgment very strongly favor one over the other. Its importance is demonstrated in practice.
9	Absolutely more important	The evidence favoring one over the other is of the highest possible validity.
2, 4, 6, 8	Intermediate values	When compromise is needed.

Table 1

A basic, but very reasonable, assumption is that if attribute *A* is absolutely more important than attribute *B* and is rated at 9, then *B* must be absolutely less important than *A* and is valued at 1/9.

These pair wise comparisons are carried out for all factors to be considered, usually not more than 7, and the matrix is completed. The matrix is of a very particular form which neatly supports the calculations which then ensue.

The next step is the calculation of a list of the relative *weights*, *importance*, or *value*, of the factors, such as cost and operability, which are relevant to the problem in question (technically, this list is called an ***eigenvector***). If, perhaps, cost is very much more important than operability, then, on a simple interpretation, the cheap equipment is called for though, as we shall see, matters are not so straightforward. The final stage is to calculate a ***Consistency Ratio (CR)*** to measure how consistent the judgments have been relative to large samples of purely random judgments. If the ***CR*** is much in excess of 0.1 the judgments are untrustworthy because they are too close for comfort to randomness and the exercise is valueless or must be repeated. It is easy to make a minimum number of judgments after which the rest can be calculated to enforce a perhaps unrealistically perfect consistency.

The AHP is sometimes sadly misused and the analysis stops with the calculation of the eigenvector from the pair wise comparisons of relative importance (sometimes without even computing the CR!) but the AHPs true subtlety lies in the fact that it is, as its name says, a Hierarchy process. The first eigenvector has given the relative importance attached to requirements, such as cost and reliability, but different machines contribute to differing extents to the satisfaction of those requirements. Thus, subsequent matrices can be developed to show how X, Y and Z respectively satisfy the needs of the firm. (The matrices from this lower level in the hierarchy will each have their own eigenvectors and CRs.) The final step is to use standard matrix calculations to produce an overall vector giving the answer we seek, namely the relative merits of X, Y and Z vis-à-vis the firms requirements.

Some theory

Consider n elements to be compared, $C_1 \dots C_n$ and denote the relative weight (or priority or significance) of C_i with respect to C_j by a_{ij} and form a square matrix $A=(a_{ij})$ of order n with the constraints that $a_{ij} = 1/a_{ji}$, for $i \neq j$, and $a_{ii} = 1$, all i. Such a matrix is said to be a ***reciprocal matrix***. The weights are consistent if they are transitive, that is $a_{ik} = a_{ij}a_{jk}$ for all i, j, and k. Such a matrix might exist if the a_{ij} are calculated from exactly measured data. Then find a vector ω of order n such that $A\omega = \lambda\omega$. For such a matrix, ω is said to be an ***eigenvector*** (of order n) and λ is an ***eigenvalue***. For a consistent matrix, $\lambda = n$. For matrices involving human judgment, the condition $a_{ik} = a_{ij}a_{jk}$ does not hold as human judgments are inconsistent to a greater or lesser degree. In such a case the ω vector satisfies the equation $A\omega= \lambda_{max}\omega$ and $\lambda_{max} \geq n$. The difference, if any, between λ_{max} and n is an indication of the inconsistency of the judgments. If $\lambda_{max} = n$ then the judgements have turned out to be consistent. Finally, a ***Consistency Index*** can be calculated from $(\lambda_{max}-n)/(n-1)$. That needs to be assessed against judgments made completely at random and Saaty has calculated large samples of random matrices of increasing order and the Consistency Indices of those matrices. A true ***Consistency Ratio*** is calculated by dividing the ***Consistency Index*** for the set of judgments by the ***Index*** for the corresponding random matrix. Saaty suggests that if that ratio exceeds 0.1 the set of judgments may be too inconsistent to be reliable. In practice, CRs of more than 0.1 sometimes have to be accepted. If CR equals 0 then that means that the judgments are perfectly consistent.

Some mathematics

There are several methods for calculating the eigenvector. Multiplying together the entries in each row of the matrix and then taking the n^{th} root of that product gives a very good approximation to the correct answer. The n^{th} roots are summed and that sum is used to normalize the eigenvector elements to add to 1.00. In ***Table 2*** below, the 4^{th} root for the first row is 0.293 and that is divided by 5.024 to give 0.058 as the first element in the eigenvector. The table below gives a worked example in terms of four attributes to be compared which, for simplicity, we refer to as A, B, C, and D.

	A	B	C	D	n^{th} root of products of values	Eigenvector
A	1	1/3	1/9	1/5	0.293	0.058
B	3	1	1	1	1.316	0.262
C	9	1	1	3	2.279	0.454
D	5	1	1/3	1	1.136	0.226
Totals					5.024	1.000

Table 2

The eigenvector of the relative importance or value of A, B, C and D is (0.058, 0.262, 0.454, 0.226). Thus, C is the most valuable, B and D are behind, but roughly equal and A is very much less significant.

The next stage is to calculate λ_{max} so as to lead to the Consistency Index and the Consistency Ratio.

We first multiply on the right the matrix of judgments by the eigenvector, obtaining a new vector. The calculation for the first row in the matrix is:

$$1\times0.058 + 1/3\times0.262 + 1/9\times0.454 + 1/5\times0.226 = 0.240$$

and the remaining three rows give 1.116, 1.916 and 0.928. This vector of four elements (0.240, 1.116, 1.916, 0.928) is, of course, the product $A\omega$ and the AHP theory says that $A\omega=\lambda max\omega$ so we can now get four estimates of λ_{max} by the simple expedient of dividing each component of (0.240, 1.116, 1.916, 0.928) by the corresponding eigenvector element. This gives 0.240/0.058 = 4.137 together with 4.259, 4.22 and 4.11. The mean of these values is 4.18 and that is our estimate for λ_{max}. If any of the estimates for λ_{max} turns out to be less than n, or 4 in this case, there has been an error in the calculation, which is a useful sanity check.

The Consistency Index for a matrix is calculated from $(\lambda_{max}-n)/(n-1)$ and, since $n=4$ for this matrix, the *CI* is 0.060. The final step is to calculate the Consistency Ratio for this set of judgments using the *CI* for the corresponding value from large samples of matrices of purely random judgments using the ***Table 3*** below, derived from Saaty's book, in which the upper row is the order of the random matrix, and the lower is the corresponding index of consistency for random judgments.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.00	0.00	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49	1.51	1.48	1.56	1.57	1.59

Table 3

For this example, that gives 0.060/0.90=0.0677. Saaty argues that a *CR > 0.1* indicates that the judgments are at the limit of consistency though *CRs* > 0.1 (but not too much more) have to be accepted sometimes. In this instance, we are on safe ground. A *CR* as high as, say, 0.9 would mean that the pair wise judgments are just about random and are completely untrustworthy.